Almost-full transversals in equi-n-squares

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Joint work with D. Chakraborti, M. Christoph, Z. Hunter, R. Montgomery

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- For example, we would like to say that e^x grows faster than $100x^{100}$ and Big O notation enables us to do that.
- There are five symbols used: O, o, Ω, ω, Θ.
- For this talk, it (mostly) suffices to know that $o(1)$ means something which goes to 0 as $n \to \infty$.

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- Does every Latin square of order n have a full transversal?
- No - consider the Latin square of order 2.

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- Then,

$$\sum_{g \in \mathbb{Z}_{2m}} g = \sum_{x \in T} entry(x) = \sum_{x \in T} (row(x) + column(x)) = 2 \cdot \sum_{g \in \mathbb{Z}_{2m}} g.$$

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- However, as

$$\sum_{g \in \mathbb{Z}_{2m}} g = m$$

in $\mathbb{Z}_{2m}$, we obtain a contradiction.

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- In 2022, Keevash, Pokrovskiy, Sudakov and Yepremyan improved this bound to $n - O(\log n / \log \log n)$.

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Theorem (Montgomery 2023+)

There is some $n_0 \in \mathbb{N}$ such that every Latin square of order $n \geq n_0$ contains a transversal of size $n-1$.

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- It was further improved in 2024+ by Anastos and Morris to $(3/4 - o(1))n$.

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Question (Pokrovskiy and Sudakov 2019)

In any equi-n-square, is there always a transversal of size $(1-o(1))n$?

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Theorem 1.1 (Chakraborti, Christoph, Hunter, Montgomery and P. 2024+)

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Theorem 1.2 (Chakraborti, Christoph, Hunter, Montgomery and P. 2024+)

For each $n \in \mathbb{N}$, there is an equi-n-square with no transversals of size $n - \left(\frac{1}{2\sqrt{2}} + o(1)\right)\sqrt{n}$.

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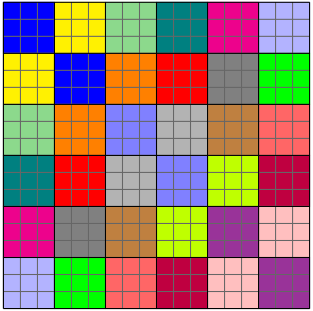

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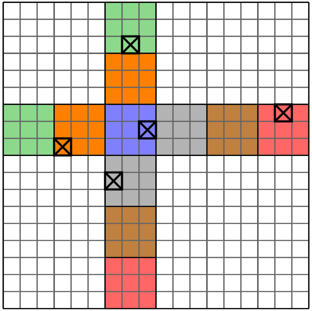

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Fix any $\varepsilon > 0$. Let S be an equi-n-square for which each symbol appears at most $o(n)$ times in every row and column. Then S has a transversal of size $(1-\varepsilon)n$.

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Theorem (Rödl's Nibble)

Fix $\varepsilon > 0$ and $r \in \mathbb{N}$, the following holds for sufficiently large m and d. Let H be an r-uniform, d-regular, m-vertex hypergraph with every pair of vertices u, v satisfying $d(u, v) \leq o(m)$. Then H has a matching with $(1-\varepsilon)m/r$ edges.

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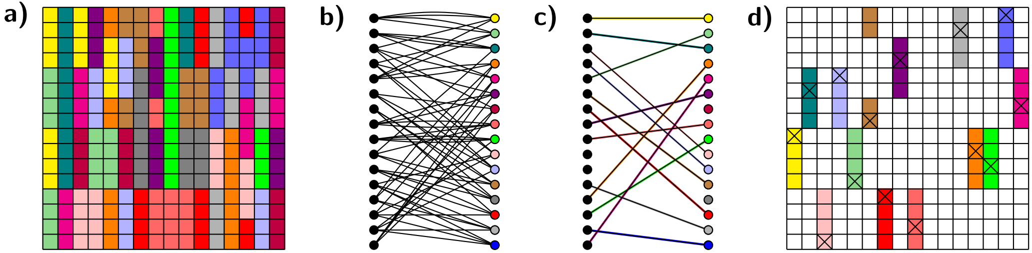

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Conjecture (Chakraborti, Christoph, Hunter, Montgomery and P. 2024+)

There exists $C > 0$ such that every equi-n-square has a transversal of size at least $n - C\sqrt{n}$.

- Thank you for listening!